Riemann Integral $f : [a,b] \to [m, M]$ (real $m<M$)

$$\alpha = \underline{\int} f = \sup\{u(P) : P \in par[a,b]\}, \text{ where}$$

$$u(P) := \sum_{i=1}^{n} m_i \, \ell(I_i), \quad \forall P = \{I_1, \cdots, I_n\}$$

...

equivalently, can be dropped → except at partition pts

$$\beta = \sup\left\{ \int g : g \text{ step function on } [a,b] \text{ s.t. } g \le f \text{ on } [a,b] \right\}$$

where $\int g := \sum_{i=1}^{n} c_i \, \ell(I_i)$ where $\{I_1, \cdots, I_n\} \in par[a,b]$

and $g = c_i$ on each $\overset{\circ}{I_i}$
$\le f$ ... (interior)

$$\gamma = \lim_{P} u(P)$$

$$\xi = \lim_{\|P\| \to 0} u(P)$$

Then $\alpha = \beta = \gamma = \xi \in [m(b-a), M(b-a)]$

Hints:

1. α always exists and $\alpha = \beta$
2. Let $\varepsilon > 0$. Then $\exists \, P_\varepsilon \in par[a,b]$ s.t. $\alpha - \varepsilon < u(P_\varepsilon) \le \alpha$.

For all $P \supseteq P_\varepsilon$ one has $\alpha - \varepsilon < u(P_\varepsilon) \le u(P) \le \alpha$.

This implies that $\lim_{P} u(P) = \alpha$.

3. Let $\varepsilon > 0$ and take P_ε as above; let

$$P_\varepsilon := \{I_1, I_2, \cdots I_N\} \text{ with some } N \in \mathbb{N}.$$

Let $\delta > 0$ be such that

$$N\delta(M-m) < \varepsilon;$$

let $P \in \text{par}[a,b]$ with $\|P\| < \delta$. By #4 below

$$(*) \quad u(P \cup P_\varepsilon) - u(P) \leq N(M-m)\|P\| \left(\leq N\delta(M-m) < \varepsilon\right)$$

and it follows that

$$u(P_\varepsilon) \leq u(P \cup P_\varepsilon) < u(P) + \varepsilon$$

and so

$$(u(P) \leq)\ \alpha < \varepsilon + u(P_\varepsilon) < u(P) + 2\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this means that $\lim\limits_{\|P\| \to 0} u(P) = \alpha$.

4. Let $P', P \in \text{par}[a,b]$, and let $P \subseteq P' = P \cup \{\xi\}$ (say $x_{i-1} < \xi < x_i$ for some $i \in \{1, 2 \ldots n\}$, $P := \{x_0, x_1, \cdots, x_n\}$)

i.e. $\xi \in I_i^\circ$

Then

$$(**) \quad u(P') - u(P) \leq (M-m)\|P\|$$

(so the required inequality in $(*)$ holds for $N = 1$, and so for any $N \in \mathbb{N}$ by repeating the same).

Write $I_i = I_i' \cup I_i''$ (with ξ being the right-end and left-end resp.) and let m_i', m_i'', M_i', M_i'' be defined correspondingly

Then

$$u(P') - u(P) = \left[m_1' \, \ell(I_i') + m_1'' \, \ell(I_i'')\right] - \left[m_i \ell(I_1') + m_i \, \ell(I_i'')\right]$$

$$= (m_1' - m_1) \, \ell(I_i') + (m_i'' - m_1) \, \ell(I_i'')$$

$$\leq (M-m) \, \ell(I_1) \leq (M-m) \, \|P\|.$$

5. $f \mapsto \underline{\int} f$ is "super-additive" $\left(\underline{\int}(f_1+f_2) \geq \underline{\int} f_1 + \underline{\int} f_2\right)$

$f \mapsto \overline{\int} f$ is "subadditive" $\left(\overline{\int}(f_1+f_2) \leq \overline{\int} f_1 + \overline{\int} f_2\right)$

for all bounded functions.

6. Use the upper sums to define $\alpha', \beta', \gamma', \xi'$ and establish the corresponding results.

(f is Riemann integrable if $\alpha = \alpha'$)

7. f is Riemann integrable iff

$\eta := \lim_{P} s(P; \xi_1, \cdots \xi_n)$ exists in $\mathbb{R}$

& iff $\zeta = \lim_{\|P\| \to 0} s(\text{------})$ exists in $\mathbb{R}$.

(& then all equal: $\alpha = \alpha' = \eta = \zeta$)